TIME VARYING MAGNETIC FIELDS AND MAXWELL'S EQUATIONS

Introduction

Electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying current.

- Stationary charges → Electrostatic fields
- Steady current → Magnetostatic fields
- Time-varying current → Electromagnetic fields (or waves)

Faraday discovered that the induced emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit

This is called Faraday's Law, and it can be expressed as

$$V_{emf} = -\frac{d\text{ª}}{dt} = -N\frac{d\Psi}{dt} \qquad 1.1$$

where N is the number of turns in the circuit and ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as Lenz's Law, and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic filed produced by the induced current will oppose the original magnetic field.

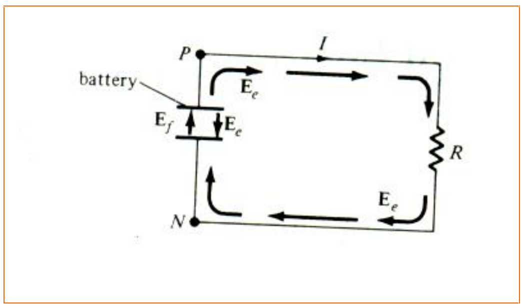

Fig. 1 A circuit showing emf-producing field E_f and electrostatic field E_e

TRANSFORMER AND MOTIONAL EMFS

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single (N = 1), eq. (1.1) becomes

$$V_{emf} = -N\frac{d\Psi}{dt} \qquad 1.2$$

In terms of **E** and **B,** eq. (1.2) can be written as

$$V_{emf} = \oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot dS \qquad 1.3$$

where, ψ has been replaced by $\int_S B \cdot dS$ and S is the surface area of the circuit bounded by the closed path L. It is clear from eq. (1.3) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that d**l** and d**S** in eq. (1.3) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in Figure 2. The variation of flux with time as in eq. (1.1) or eq. (1.3) may be caused in three ways:

1. By having a stationary loop in a time-varying **B** field
2. By having a time-varying loop area in a static **B** field
3. By having a time-varying loop area in a time-varying **B** field.

A. STATIONARY LOOP IN TIME-VARYING B FIELD (TRANSFORMER EMF)

This is the case portrayed in Figure 2 where a stationary conducting loop is in a time varying magnetic **B** field. Equation (1.3) becomes

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS \qquad 1.4$$

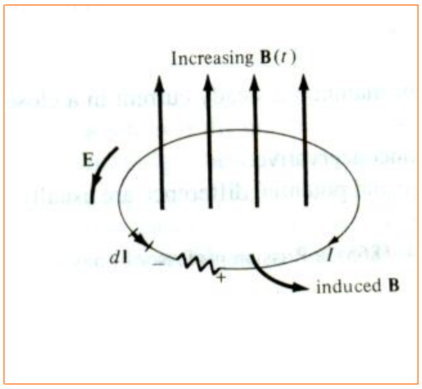

Fig. 2: Induced emf due to a stationary loop in a time varying **B** field.

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (1.4), we obtain

$$\int_S (\nabla \times E)\cdot dS = -\int_S \frac{\partial B}{\partial t}\cdot dS \qquad 1.5$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad 1.6$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time varying E field is not conservative (∇ x E $\neq$ 0). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field.

B. MOVING LOOP IN STATIC B FIELD (MOTIONAL EMF)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. We recall from eq. (1.7) that the force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F_m} = Qu \times \mathbf{B} \qquad 1.7$$

We define the *motional electric field* E_m as

$$E_m = \frac{F_m}{Q} = u \times B \qquad 1.8$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L E_m \cdot dl = \oint_L (u \times B)\cdot dl \qquad 1.9$$

This type of emf is called *motional emf or flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.

C. MOVING LOOP IN TIME-VARYING FIELD

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining equation 1.4 and 1.9 gives the total emf as

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_L (u \times B) \cdot dl \qquad 1.10$$

$$\nabla \times E_m = \nabla \times (u \times B) \qquad 1.11$$

or from equations 1.6 and 1.11.

$$\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B) \qquad 1.12$$

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad 1.13$$

But the divergence of the curl of any vector field is identically zero.

Hence,

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J \qquad 1.14$$

The continuity of current requires that

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad 1.15$$

Thus eqs. 1.14 and 1.15 are obviously incompatible for time-varying conditions. We must modify eq. 1.13 to agree with eq. 1.15. To do this, we add a term to eq. 1.13, so that it becomes

$$\nabla \times H = J + J_d \qquad 1.16$$

where J_d is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J + \nabla \cdot J_d \qquad 1.17$$

In order for eq. 1.17 to agree with eq. 1.15,

$$\nabla \cdot J_d = -\nabla \cdot J = \frac{\partial ^{\circ}{}_{v}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \frac{\partial D}{\partial t} \qquad 1.18$$

or

$$J_d = \frac{\partial D}{\partial t} \qquad 1.19$$

Substituting eq. 1.19 into eq. 1.15 results in

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad 1.20$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $J_d = \partial D/\partial t$ is known as *displacement current density and* J is the conduction current density ($J = \sigma E$)[3].

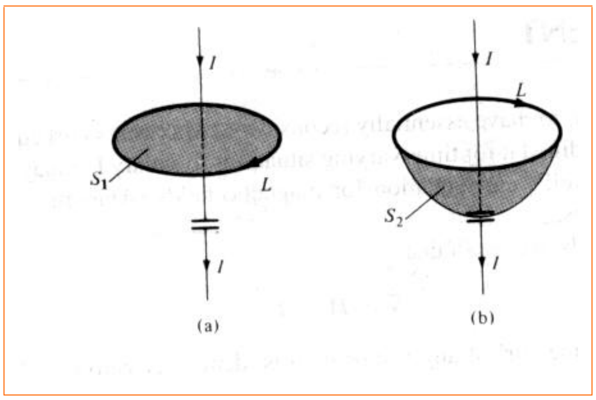

Fig. 3 Two surfaces of integration showing the need for J_d in Ampere's circuit law

The insertion of J_d into eq. 1.13 was one of the major contribution of Maxwell. Without the term J_d, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, J_d is usually neglected compared with J. however, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. 1.20 could not be verified experimentally.

Based on displacement current density, we define the displacement current as

$$I_d = \int J_d \cdot dS = \int \frac{\partial D}{\partial t} \cdot dS \qquad 1.21$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is that through a capacitor when an alternating voltage source is applied to its plates.

PROBLEM: A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^3 t V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\,\varepsilon_0$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S\frac{d\rho_s}{dt} = S\frac{dD}{dt} = \varepsilon S\frac{dE}{dt} = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5\times10^{-4}}{3\times10^{-3}} \cdot 10^3 \times 50\cos 10^3 t$$

$= 147.4 \cos 10^3\, t$ nA

EQUATION OF CONTINUITY FOR TIME VARYING FIELDS

Equation of continuity in point form is

$$\nabla \,.\, J = -\rho_v$$

where,

J = conduction current density (A/M^2)

Pυ = volume charge density (C/M^3), $\circ_v = \frac{\partial \circ_v}{\partial t}$

∇ = vector differential operator (1/m)

$$\nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

Proof: Consider a closed surface enclosing a charge *Q*. There exists an outward flow of current given by

$$I = \oint_S J \cdot dS$$

This is equation of continuity in **integral form.**

From the principle of conservation of charge, we have

$$I = \oint_S J \cdot dS = \frac{-dQ}{dt}$$

From the divergence theorem, we have

$$I = \oint_S J \cdot dS = \int_v (\nabla \cdot J) d^3$$

Thus, $$\int_3 (\nabla \cdot J) d^3 = \frac{-dQ}{dt}$$

By definition, $Q = \int_v \rho_v \, dv$

where, ρ_v = volume charge density (C/m^3)

So, $\int_v (\nabla \cdot J) dv = \int \frac{\partial \rho_v}{\partial t} dv = \int - \dot{\rho}_v \, dv$

where $\dot{\rho}_v = \frac{\partial \rho_v}{\partial t}$

The volume integrals are equal only if their integrands are equal.

Thus, $\nabla \cdot \mathbf{J} = - \dot{\rho}_v$

MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S D \cdot dS = \int_v \rho_v dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S B \cdot dS = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot dl = -\frac{\partial}{\partial t} \int_s B \cdot dS$	Faraday's Law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_s J \cdot dS$	Ampere's circuit law

MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS

These are basically four in number.

Maxwell's equations in **differential form** are given by

$\nabla \times H = \frac{\partial D}{\partial t} + J$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla . D = \rho_{\upsilon}$

$\nabla . B = 0$

Here,

H = magnetic field strength (A/m)

D = electric flux density, (C/m^2)

($\partial D/\partial t$) = displacement electric current density (A/m^2)

J = conduction current density (A/m^2)

E = electric field (V/m)

B = magnetic flux density wb/m^2 or Tesla

($\partial B/\partial t$) = time-derivative of magnetic flux density (wb/m^2 -sec)

B is called magnetic current density (V/m^2) or Tesla/sec

P_{υ} = volume charge density (C/m^3)

Maxwell's equations for time varying fields in **integral form** are given by

$$\oint_L H \cdot dL = \int_S \left(\dot{D} + J \right) \cdot dS$$

$$\oint_L E \cdot dL = -\int_S B \cdot dS$$

$$\oint_S D \cdot dS = \oint_3 \circ_3 d^3$$

$$\oint_S B \cdot dS = 0$$

MEANING OF MAXWELL'S EQUATIONS

1. The first Maxwell's equation states that the magnetomotive force around a closed path is equal to the sum of electric displacement and, conduction currents through any surface bounded by the path.

2. The second law states that the electromotive force around a closed path is equal to the inflow of magnetic current through any surface bounded by the path.

3. The third law states that the total electric displacement flux passing through a closed surface (Gaussian surface) is equal to the total charge inside the surface.

4. The fourth law states that the total magnetic flux passing through any closed surface is zero.

MAXWELL'S EQUATIONS FOR STATIC FIELDS

Maxwell's Equations for static fields are:

$$\nabla \times H = J \leftrightarrow \oint_L H \cdot dL = \int_S J \cdot dS$$

$$\nabla \times E = 0 \leftrightarrow \oint_L E \cdot dL = 0$$

$$\nabla \cdot D = {}^\circ{}_3 \leftrightarrow \oint_S D \cdot dS = \oint_3 {}^\circ{}_3 \, d^3$$

$$\nabla \cdot B = 0 \leftrightarrow \oint_S B \cdot dS = 0$$

As the fields are static, all the field terms which have time derivatives are zero, that is, $\frac{\partial D}{\partial t} = 0$, $\frac{\partial B}{\partial t} = 0$.

PROOF OF MAXWELLS EQUATIONS

1. From Ampere's circuital law, we have

$$\nabla \times H = J$$

Take dot product on both sides

$$\nabla \cdot \nabla \times H = \nabla \cdot J$$

As the divergence of curl of a vector is zero,

$$RHS = \nabla \cdot J = 0$$

But the equation of continuity in point form is

$$\nabla \cdot J = \frac{-\partial {}^\circ{}_3}{\partial t} = -{}^\circ{}_3$$

This means that if $\nabla \times H = J$ is true, it is resulting in $\nabla . J = 0$.

As the equation of continuity is more fundamental, Ampere's circuital law should be modified. Hence we can write

$$\nabla \times H = J + F$$

Take dot product on both sides

$$\nabla . \nabla \times H = \nabla . J + \nabla . F$$

that is, $\nabla . \nabla \times H = 0 = \nabla . J + \nabla . F$

Substituting the value of $\nabla.J$ from the equation of continuity in the above expression, we get

$$\nabla . F + (-\rho_\upsilon) = 0$$

or, $\nabla . F = -\rho_\upsilon$

The point form of Gauss's law is

$$\nabla . D = \rho_\upsilon$$

or, $\nabla . D = \rho_\upsilon$

From the above expressions, we get

$$\nabla . F = \nabla . D$$

The divergence of two vectors are equal only if the vectors are identical,

that is, $F = D$

So, $\nabla \times H = D + J$

Hence proved.

2. According to Faraday's law,

$$emf = \frac{-d¤}{dt}$$

ϕ = magnetic flux, (wb)

and by definition,

$$emf = \oint_L E \cdot dL$$

$$\oint_L E \cdot dL = \frac{-d¤}{dt}$$

But $$¤ = \int_S B \cdot dS$$

$$\oint_L E \cdot dL = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

$$= -\int_S B \cdot dS, \quad B = \frac{\partial B}{\partial t}$$

Applying Stoke's theorem to LHS, we get

$$\oint_L E \cdot dL = -\int_S (\nabla \times E) \cdot dS$$

$$\int_S (\nabla \times E) \cdot dS = \int_S - B \cdot dS$$

Two surface integrals are equal only if their integrands are equal,

that is, ∇ x E = - B

Hence proved.

3. From Gauss's law in electric field, we have

$$\oint_S D \cdot dS = Q = \int_3 {}^\circ{}_3\, d^3$$

Applying divergence theorem to LHS, we get

$$\oint_S D \cdot dS = \int_3 (\nabla \cdot D) d^3 = \int_3 {}^\circ{}_3\, d^3$$

Two volume integrals are equal if their integrands are equal,

that is, $\nabla \, . \, D = \rho_\upsilon$

Hence proved.

4. We have Gauss's law for magnetic fields as

$$\oint_S B \cdot dS = 0$$

RHS is zero as there are no isolated magnetic charges and the magnetic flux lines are closed loops.

Applying divergence theorem to LHS, we get

$$\oint_3 \nabla \cdot B \; d^3 = 0$$

or,

$\nabla \, . \, B = 0$ Hence proved.

PROBLEM 1:

Given $E = 10 \sin(\omega t - \beta y)\, a_y$ V/m, in free space, determine D, B and H.

Solution:

$E = 10 \sin(\omega t - \beta y)\, a_y$, V/m

$D = \epsilon_0 E$, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$D = 10\epsilon_0 \sin(\omega t - \beta y)\, a_y$, C/m^2

Second Maxwell's equation is

$\nabla \times E = -B$

That is,
$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

or,
$$\nabla \times E = a_x\left[-\frac{\partial}{\partial z}E_y\right] + 0 + a_z\left[\frac{\partial}{\partial x}E_y\right]$$

As $E_y = 10 \sin(\omega t - \beta z)$ V/m

$$\frac{\partial E_y}{\partial x} = 0$$

Now, $\nabla \times E$ becomes

$$\nabla \times E = -\frac{\partial E_y}{\partial z} a_x$$

$= 10\, \beta \cos(\omega t - \beta z)\, a_x$

$$= -\frac{\partial B}{\partial t}$$

$$B = -\int 10 \;\; \cos(\omega t - \;\; z) dt \;\; a_x$$

or
$$B = \frac{10}{\omega}\sin(\omega t - \;\; z) a_z, wb/m^2$$

and
$$H = \frac{B}{\mu_0} = \frac{10}{\mu_0 \omega}\sin(\omega t - \;\; z) a_z, A/m$$

PROBLEM 2: If the electric field strength, E of an electromagnetic wave in free space is given by E = 2 cos ω $\left(t - \frac{z}{3_0}\right) a_y$ V/m, find the magnetic field, H.

Solution: We have

∂B/∂t = -∇ x E

$$= \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\left[a_x \left[-\frac{\partial}{\partial z} E_y \right] + a_y (0) + a_z \left[\frac{\partial}{\partial x} E_y \right] \right]$$

$$= \frac{\partial E_y}{\partial z} a_x$$

$$= \frac{2\mu}{3_0} \sin \mu \left(t - \frac{z}{3_0} \right) a_x$$

$$B = \frac{2\mu}{3_0} \int \sin \mu \left(t - \frac{z}{3_0} \right) dt \; a_x$$

or, $$B = \frac{-2\mu}{3_0 \mu} \cos \mu \left(t - \frac{z}{3_0} \right) a_x$$

or, $$H = \frac{B}{«_0} = \frac{-2}{3_0 «_0} \cos \mu \left(t - \frac{z}{3_0} \right) a_x \qquad ¦_0 = \sqrt{\frac{«_0}{\in_0}} = 120®\Omega$$

Thus, $$H = \frac{-2}{¦_0} \cos \mu \left(t - \frac{z}{3_0} \right) a_x \qquad \left[3_0 = \frac{1}{\sqrt{«_0 \in_0}} \right]$$

$$H = \frac{-1}{60®} \cos \mu \left(t - \frac{z}{3_0} \right) a_x A/m$$

PROBLEM 3: If the electric field strength of a radio broadcast signal at a TV receiver is given by

$$E = 5.0 \cos(\omega t - \beta y)\, a_z, \text{ V/m},$$

determine the displacement current density. If the same field exists in a medium whose conductivity is given by 2.0 x 10^3 (mho)/cm, find the conduction current density.

Solution:

E at a TV receiver in free space

$= 5.0 \cos(\omega t - \beta y)\, a_z$, V/m

Electric flux density

$D = \in_0 E = 5 \in_0 \cos(\omega t - \beta y)\, a_z$, V/m

The displacement current density

$$J_d = D = \frac{\partial D}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[-5 \in_0 \cos(\mu t - \ y) a_z\right]$$

$J_d = -5\in_0 \omega \sin(\omega t - \beta y)\, a_z$, V/$m^2$

The conduction current density,

$J_c = \sigma E$

$\sigma = 2.0 \times 10^3$ (mho) /cm

$= 2 \times 10^5$ mho /m

$J_c = 2 \times 10^5 \times 5 \cos(\omega t - \beta y)\, a_z$

$\boldsymbol{J_c = 10^6 \cos(\mu t - \ y)\, a_z}$ **V/$\boldsymbol{m^2}$**